Meta-Analysis for Functions of Heterogeneous Correlation Matrices

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Preview

- Background and motivation
  - meta-analysis (MA)
  - functions of correlations
- MA for correlation matrices
- MA for functions of correlations
  - conventional method
  - proposed methods
- Monte Carlo study
- Conclusions

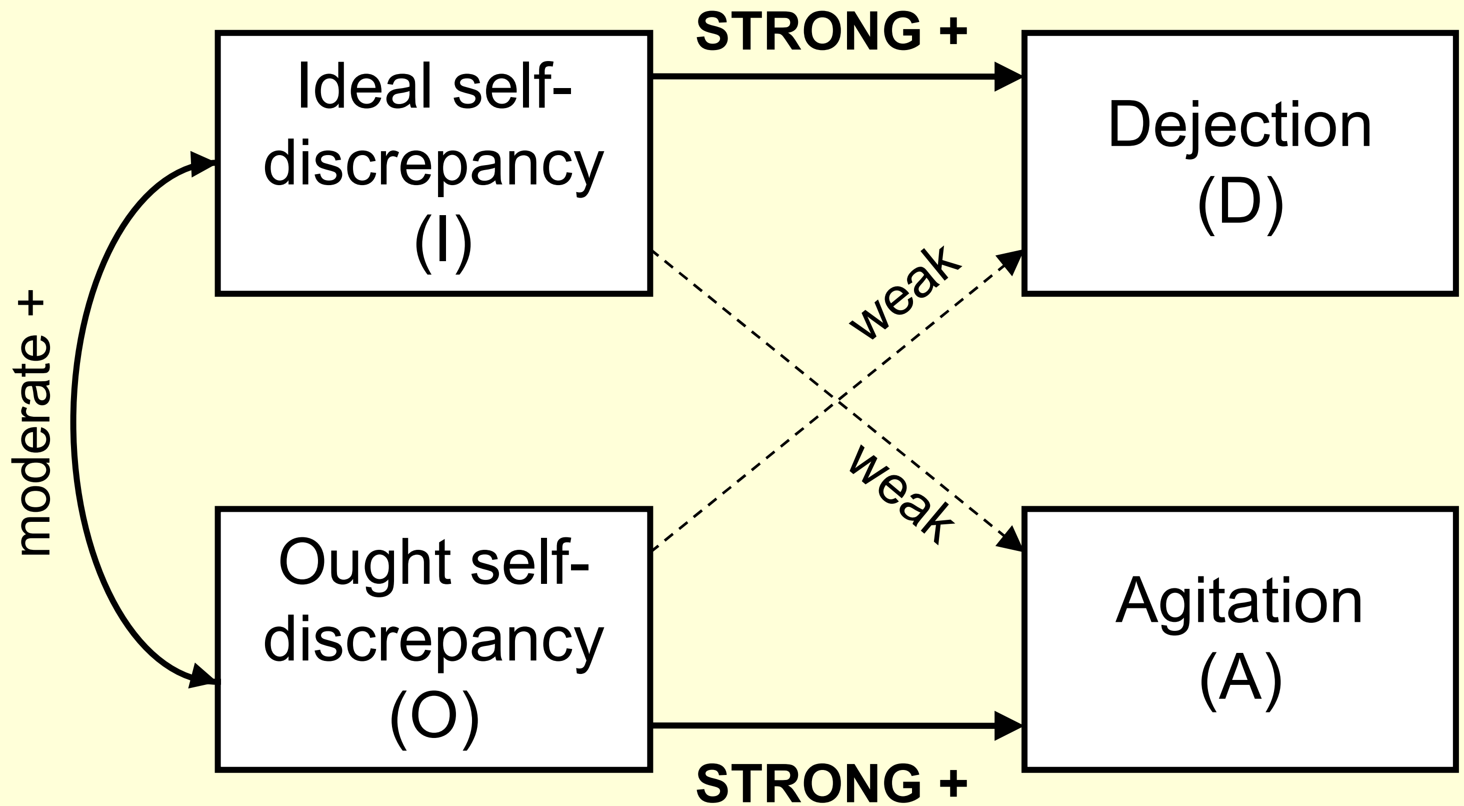
Background and Motivation

- Meta-analysis (MA)
  - techniques to compare and combine studies' results
  - "results" (data) are measures of effect magnitude
    - correlation, (standardized) mean difference, odds ratio
    - transformations of these (e.g., to stabilize variance)
    - multivariate versions of these (e.g., correlation matrix)
  - typical aims: estimate and make inferences about effects' average, dispersion, or covariates
  - two popular inference models
    - fixed effects (FE): all studies share common effect parameter
    - random effects (RE): effect parameter varies among studies

Background and Motivation

- Functions of (product-moment) correlations
  - partial, (squared) multiple, or canonical correlations
  - coefficients in regression, path, or factor models
  - determinants, eigenvalues, etc.
  - contrasts or other linear combinations of these
- MA for functions of correlations
  - pioneered by Becker (1992), for linear models
  - some refinements and assessments over 15 years
  - meta-analytic SEM (Cheung & Chan, 2005), FE case
  - rarely used by meta-analysts—no friendly software?

Self-Discrepancy Theory: Path Model



Correlations Between Two Types of Self-Discrepancy and Two Types of Emotional Distress

Study	*n*	IO	ID	OD	IA	OA	DA
1	36	—	17	9	26	38	—
2	50	—	14	11	8	2	—
3	52	59	—	—	—	—	—
4	53	43	29	15	18	32	—
5	72	—	39	26	21	43	—
6	91	36	36	—	—	—	—
7	93	23	—	—	—	—	—
8	145	—	—	—	—	—	56
9	163	48	41	8	22	34	41
10	169	—	38	—	40	—	26
11	199	56.1	24.8	16.3	—	—	—

Note. I = Ideal self-discrepancy, O = Ought self-discrepancy, A = agitation, D = dejection. Table entry is 100 × correlation.

MA for Correlation Matrices

- Notation & model, complete-data case
  - *I* independent studies with "same" MVN variables
  - let θ_i be study *i*'s vector of *J* Pearson-*r* ($\theta = \rho$) or Fisher-*z* ($\theta = \zeta$) correlation parameters, $i = 1, ..., I$
  - let $\mathbf{t}_i$ be study *i*'s vector of *J* Pearson-*r* ($\mathbf{t} = \mathbf{r}$) or Fisher-*z* ($\mathbf{t} = \mathbf{z}$) sample correlations from n_i subjects
  - RE model, no covariates (Becker, 1992):

$$\mathbf{t}_i = \theta_i + \mathbf{e}_i = \mu_\theta + \mathbf{u}_i + \mathbf{e}_i ,$$

    where $\mathbf{u}_i \sim N_J(0, \mathrm{T}_\theta)$ independent of $\mathbf{e}_i \sim N_J(0, \mathbf{V}_{\theta i})$, $\forall\ i$
  - $\mathbf{V}_{\theta i}$ known, but use Olkin and Siotani's (1976) asymptotic approximation $\mathbf{V}_\theta(\rho_i, n_i)$ with estimate of ρ_i

MA for Correlation Matrices

- Estimation of and inference about μ_θ
  - with T_θ known, estimate μ_θ by GLS:

    $$\hat{\mu}_\theta = (\mathbf{X}^\mathrm{T}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^\mathrm{T}\mathbf{W}\mathbf{t}$$

    where $\mathbf{X}$ is I stacked identity matrices $\mathbf{I}_J$; $\mathbf{W}$ is block-diagonal with blocks $(\mathrm{T}_\theta + \mathbf{V}_{\theta 1})^{-1}$, ..., $(\mathrm{T}_\theta + \mathbf{V}_{\theta I})^{-1}$; and data $\mathbf{t}$ is stacked $\mathbf{t}_1$, ..., $\mathbf{t}_I$
  - for missing elements of $\mathbf{t}_i$, delete rows and columns
  - for normal-theory inference about μ_θ use

    $$\mathrm{C\hat{o}v}(\hat{\mu}_\theta) = (\mathbf{X}^\mathrm{T}\mathbf{W}\mathbf{X})^{-1}$$
  - MLE $\hat{\mathrm{T}}_\theta$ by EM algorithm (Becker & Schram), which assumes $\theta_i \sim N_J(\mu_\theta, \mathrm{T}_\theta)\ \forall\ i$—yields EB estimate of θ_i

MA for Functions of Correlations

- Notation
  - define $\gamma_i \equiv g(\theta_i)$ and $\mathbf{g}_i \equiv g(\mathbf{t}_i)$, a K-component function g applied to study i's correlation vectors θ_i and $\mathbf{t}_i$
  - let $\mu_\gamma \equiv \mathrm{E}(\gamma_i)$ and $\mathrm{T}_\gamma \equiv \mathrm{Cov}(\gamma_i)$
- Conventional approach (Becker, 1992)
  - standardized coefficients in univariate linear model
  - FE estimator of $\gamma \equiv g(\rho)$ with delta-method inference:

$$\hat{\gamma} = g(\hat{\rho})$$

$$\mathrm{C\hat{o}v}(\hat{\gamma}) = \mathbf{A}_{\hat{\gamma},\hat{\rho}} \mathrm{C\hat{o}v}(\hat{\rho}) \mathbf{A}_{\hat{\gamma},\hat{\rho}}^{\mathrm{T}} ,$$

where $\mathbf{A}_{\hat{\gamma},\hat{\rho}}$ is Jacobian matrix of $\hat{\gamma}$ wrt $\hat{\rho}$ at $\hat{\rho} \approx \mathrm{E}(\hat{\rho})$

MA for Functions of Correlations

- Conventional approach (Becker, 1992)
  - RE estimator $g(\hat{\mu}_\rho)$—of $g(\mu_\rho)$? of μ_γ?—with delta-method inference:

$$\hat{\mathrm{Cov}}[g(\hat{\mu}_\rho)] = \mathbf{A}_{g(\hat{\mu}_\rho),\hat{\mu}_\rho} \hat{\mathrm{Cov}}(\hat{\mu}_\rho) \mathbf{A}^{\mathrm{T}}_{g(\hat{\mu}_\rho),\hat{\mu}_\rho}$$

  - advantages
    - easy to implement for many functions, especially with numerical derivatives in Jacobian matrix
    - FE and RE use same technique and derivative expressions
  - limitations in RE case
    - often "natural" estimand $g(\mu_\rho) \equiv g[\mathrm{E}(\rho)] \neq \mathrm{E}[g(\rho)] \equiv \mu_\gamma$
    - no strategy to estimate T_γ or other features of γ distribution
    - unclear how to use Fisher-*z* analysis: $g[\tanh(\hat{\mu}_\zeta)]$?

MA for Functions of Correlations

- Proposal 0: Direct meta-analysis of $\mathbf{g}_i = g(\mathbf{t}_i)$
  - obtain $\mathbf{V}_{\gamma i} \equiv \mathrm{Cov}(\mathbf{g}_i)$ from $\mathbf{V}_{\theta i}$ by delta method
  - same random-effects model for $\mathbf{g}_i$ as for $\mathbf{t}_i$:

$$\mathbf{g}_i = \gamma_i + \mathbf{e}_i = \mu_\gamma + \mathbf{u}_i + \mathbf{e}_i$$

  - estimate T_γ by EM and μ_γ and $\mathrm{Cov}(\hat{\mu}_\gamma)$ by GLS
  - advantages
    - all usual meta-analytic results for both μ_γ and T_γ
    - extends readily to models with study-level covariates
  - limitations
    - major: not applicable to studies with incomplete $\mathbf{t}_i$
    - minor: $\gamma_i \sim N_J(\mu_\gamma, \mathrm{T}_\gamma)$ may be difficult to justify for some g

MA for Functions of Correlations

- Proposal 1: Integral transformation (IT) with delta-method inference
  - assume $f(\theta) = N_J(\mu_\theta, \mathrm{T}_\theta)$
  - estimate $\mu_\gamma \equiv \mathrm{E}[g(\theta)] \equiv \int g(\theta) f(\theta) d\theta$ and $\mathrm{T}_\gamma \equiv \mathrm{Cov}[g(\theta)]$ as integrals (Monte Carlo), with μ_θ and T_θ estimated:
    - compute $\gamma^* = g(\theta^*)$ for M samples from $\theta^* \sim N_J(\hat{\mu}_\theta, \hat{\mathrm{T}}_\theta)$
    - $\hat{\mu}_\gamma$ and $\hat{\mathrm{T}}_\gamma$ are sample mean and covariance matrix of γ_m^*
  - delta-method inference, treating $\hat{\mathrm{T}}_\theta$ as known:

$$\mathrm{C\hat{o}v}(\hat{\mu}_\gamma) = \mathbf{A}_{\hat{\mu}_\gamma, \hat{\mu}_\theta} \mathrm{C\hat{o}v}(\hat{\mu}_\theta) \mathbf{A}_{\hat{\mu}_\gamma, \hat{\mu}_\theta}^{\mathrm{T}}$$

  - numerical derivatives of $\hat{\mu}_\gamma$ wrt $\hat{\mu}_\theta$ usually necessary

MA for Functions of Correlations

- Proposal 1: IT with delta method (continued)
  - advantages
    - "natural" estimands are μ_γ and T_γ
    - accommodates incomplete $\mathbf{t}_i$
    - easy to implement for many functions
    - supports both Pearson-*r* and Fisher-*z* analysis
  - limitations
    - inference neglects uncertainty about T_θ
    - may be sensitive to non-normality of θ
    - computationally intensive

MA for Functions of Correlations

- Proposal 2: IT with bootstrap inference
  - same integral estimators of μ_γ and T_γ as Proposal 1
  - multivariate extensions of van den Noortgate and Onghena's (2005) univariate bootstrap methods
  - strategies to obtain $\mathbf{t}_b^*$ for *B* bootstrap resamples of $\hat{\mu}_\gamma$
    - effect size: $(\mathbf{u}_{bi}^* + \mathbf{e}_{bi}^*) \sim N_J(0, \hat{\mathrm{T}}_\theta + \mathbf{V}_{\theta i})$, $\mathbf{t}_{bi}^* = \hat{\mu}_\theta + \mathbf{u}_{bi}^* + \mathbf{e}_{bi}^*$
    - raw data: $\mathbf{u}_{bi}^* \sim N_J(0, \hat{\mathrm{T}}_\theta)$, $\theta_{bi}^* = \hat{\mu}_\theta + \mathbf{u}_{bi}^*$, $\mathbf{t}_{bi}^*$ or $\tanh(\mathbf{t}_{bi}^*)$ —for ρ or ζ—from Wishart based on θ_{bi}^* and n_i
    - error: $\hat{\theta}_i = (\mathbf{V}_{\theta i}^{-1} + \hat{\mathrm{T}}_\theta^{-1})^{-1}(\mathbf{V}_{\theta i}^{-1}\mathbf{t}_i + \hat{\mathrm{T}}_\theta^{-1}\hat{\mu}_\theta)$, $\mathbf{u}_{bi}^*$ SWR from $\hat{\mathbf{u}}_i = \hat{\theta}_i - \hat{\mu}_\theta$, $\mathbf{e}_{bi}^*$ SWR from $\hat{\mathbf{e}}_i = \mathbf{t}_i - \hat{\theta}_i$, $\mathbf{t}_{bi}^* = \hat{\mu}_\theta + \mathbf{u}_{bi}^* + \mathbf{e}_{bi}^*$ (also involves rescaling and reflating residuals)
    - cases: $\mathbf{t}_{bi}^*$ SWR from $\mathbf{t}_i$

MA for Functions of Correlations

- Proposal 2: IT with bootstrap (continued)
  - with any strategy, obtain $\hat{\mu}^*_{\gamma b}$ from $\mathbf{t}^*_b$, b = 1, ..., B
  - various options for inference based on resamples
    - nonparametric confidence interval (CI), from quantiles
    - parametric CI, from covariance matrix
  - advantages
    - same as Proposal 1 (IT with delta method)
    - inferences may better reflect uncertainty about T_θ
    - may be less sensitive to violated distributional assumptions
  - limitations
    - computationally intensive
    - trouble with improper dispersion matrices

Monte Carlo Study

- Method
  - MA procedures (RE)
    - Proposal 1 (IT-DM): both *r* and *z*, M = 10,000
    - Proposal 2 (IT-B): only *z*, all 4 options, B = 200, percentile CI
  - conditions: $I \in \{10\ 20\}$, $E(n) \in \{50\ 100\}$, n_i +skewed
  - data generation (heterogeneous)
    - J = 6 correlations with $\mu_\rho = [.30, .76, .41, .28, .63, .32]^T$ and $\mathrm{T}_\rho = \mathrm{Diag}([.071^2, .033^2, .065^2, .072^2, .047^2, .070^2])$
    - $\rho_i \sim N_6(\mu_\rho, \mathrm{T}_\rho)$, then $\mathbf{r}_i$ computed from n_i 4-variate observations from $N_4(\mathbf{0}, \rho_i)$
    - 750 replications, so SE $\approx$ .015 for estimated probability
  - outcome: coverage of 95% CI for $\mu_{\gamma k} = E(\rho_{13} - \rho_{13.2})$

Monte Carlo Study

- Results (coverage proportion for 95% CI)

		r	*z*				
				IT-B			
I	E(n_i)	IT-DM	IT-DM	ES	RD	Er	Ca
10	50	.95	.94	—	—	—	—
10	100	.93	.93	—	—	—	—
20	50	.95	.93	—	—	—	—
20	100	.94	.92	.89	.88	.82	.89

IT-B strategy: ES = effect size, RD = raw data, Er = error, Ca = cases.

Conclusions

- CI performance of proposed MA methods for functions of correlations
  - IT with delta method: coverage at or somewhat below nominal, slightly better with *r* than *z*
  - IT with bootstrap: coverage markedly below nominal, clearly worst for “error” strategy
- Future work (short term)
  - improve bootstrap (e.g., larger *B*, more sophisticated inference strategy, handling of improper matrices)
  - more extensive Monte Carlo assessment (e.g., more conditions, different functions, various θ distributions)